Free Molecular Flow over a Rotating Sphere

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Nomenclature

C_D = drag coefficient
C_L = lift coefficient
F = total force acting on the sphere
$\hat{I}, \hat{J}, \hat{K}$ = unit vectors in X, Y, and Z directions, respectively
m = mass of a molecule
n_∞ = number density of the freestream molecules
p = normal component of momentum flux, Eq. (6a), or normal pressure acting on the differential area, Eq. (14a)
$\vec{q}$ = mass velocity of the freestream molecules
R = gas constant
R_0 = radius of the sphere
S = $U/(2RT)^{1/2}$, Eq. (9a)
s = $R_0\Omega/U$, Eq. (9b)
T_∞ = freestream temperature
U = $|\vec{q}|$
$\bar{u}, \bar{v}, \bar{w}$ = the x, y, and z components of the mass velocity of the freestream molecules with respect to the local coordinates, respectively
X, Y, Z = Cartesian stationary frame of reference
x, y, z = local Cartesian coordinates, fixed on the moving differential area
$\hat{x}, \hat{y}, \hat{z}$ = unit vectors in the x, y, and z directions, respectively
θ = azimuth, measured from negative X axis
ρ_∞ = mn_∞, freestream density
σ, σ' = momentum accommodation coefficients, Eq. (12) and Eq. (13)
τ = tangential component of momentum flux, Eqs. (6b) and (6c), or shearing stress acting on the differential area, Eqs. (14b) and (14c)
ϕ = latitude, measured from positive Y axis

Subscripts
I, J, K = components in the X, Y, and Z directions, respectively
i = quantity impinging to the sphere
r = quantity reflected from the sphere
w = value at sphere surface
x, y, z = components in the x, y, and z directions, respectively.

Introduction

THE determination of the aerodynamic forces acting on a satellite is a free molecular flow problem, which has been extensively discussed by Sentman.[1] The studies of the problem of a free molecular flow over a stationary sphere have been well documented.[2-5] However, the study of the problem of free molecular flow over a rotating sphere does not seem to appear, to the writer's knowledge, in the literature.

It is well known that a continuous flow passing over a rotating sphere will exert on the sphere a lift as well as a drag. In the free molecular flow, according to the present study, a "negative lift" will be exerted on the sphere.

General Expression for the Forces Acting on a Differential Surface Area of a Rotating Sphere

Consider a frame of reference which is set at the center of the sphere and is fixed in the space. The unit vectors in the X, Y and Z directions are $\hat{I}$, $\hat{J}$ and $\hat{K}$ respectively (Fig. 1). The sphere is spinning with constant angular velocity Ω about the Y axis as shown. A uniform stream of rarefied gas flows over the sphere in the X direction with mass velocity

$$\vec{q} = U\hat{I} \tag{1}$$

where U is the magnitude of the mass velocity measured with reference to the stationary frame (X, Y, Z). The Knudsen number is assumed so high that the assumptions of free molecular flow are satisfied.

A set of local coordinates (x, y, z) is fixed on the differential area, $dA = R_0^2 \sin\phi\, d\theta\, d\phi$, located at ($R_0$, θ, ϕ) on the surface of the sphere. The unit vectors in the x, y and z directions are $\hat{x}$, $\hat{y}$ and $\hat{z}$, respectively. The unit vector $\hat{x}$ is tangent to the parallel; $\hat{y}$ is pointing to the center; and $\hat{z}$ is tangent to the meridian. The unit vectors $\hat{I}$, $\hat{J}$, and $\hat{K}$ are related to the unit vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$ by the following matrix equation:

$$\begin{bmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{bmatrix} = \begin{bmatrix} \sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \\ 0 & -\cos\phi & \sin\phi \\ \cos\theta & -\sin\phi\sin\theta & -\cos\phi\sin\theta \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \tag{2}$$

The mass velocity of the molecules with respect to the local coordinates can thus be obtained as

$$\vec{q} = (U\sin\theta - R_0\Omega\sin\phi)\hat{x} + U\cos\theta\sin\phi\hat{y} + U\cos\theta\cos\phi\hat{z} \tag{3}$$

Which gives the velocity distribution function f for a molecule,

$$f = [1/(2\pi RT_\infty)^{3/2}]\exp\{-(1/2RT_\infty)[(u-\bar{u})^2 + (v-\bar{v})^2 + (w-\bar{w})^2]\} \tag{4}$$

where

$$\bar{u} = U\sin\theta - R_0\Omega\sin\phi \tag{5a}$$

$$\bar{v} = U\cos\theta\sin\phi \tag{5b}$$

$$\bar{w} = U\cos\theta\cos\phi \tag{5c}$$

T_∞ is the temperature of the freestream; R is the gas constant for the particular gas of interest; u, v and w are the x, y and z components of the total molecular velocity, respectively.

With the assumptions made for the free molecular flow, the normal and tangential components of the momentum flux impinging on the differential area under consideration are

$$p_i = \int\int\int mn_\infty v^2 f\,du\,dv\,dw \tag{6a}$$

$$\tau_{ix} = \int\int\int mn_\infty uvf\,du\,dv\,dw \tag{6b}$$

$$\tau_{iz} = \int\int\int mn_\infty wvf\,du\,dv\,dw \tag{6c}$$

where m is the mass of a molecule; n_∞ is the number density of the freestream molecules. The domain of integration for u and w is from $-\infty$ to $+\infty$, and that for v is from 0 to $+\infty$. The subscript "i" indicates the quantities impinging on the differential area.

Using Eq. (4) and Eqs. (5), we may integrate Eqs. (6) to find the normal and tangential components of the impinging momentum flux

$$p_i = (\tfrac{1}{2}\rho_\infty U^2/\pi^{1/2}S^2)\{S\cos\theta\sin\phi\exp(-S^2\cos^2\theta\sin^2\phi) + \pi^{1/2}[\tfrac{1}{2} + (S\cos\theta\sin\phi)^2][1 + \mathrm{erf}(S\cos\theta\sin\phi)]\} \tag{7a}$$

$$\tau_{ix} = (\tfrac{1}{2}\rho_\infty U^2/\pi^{1/2})F_u(\theta,\phi,s)\{(1/S)\exp[-S^2\cos^2\theta\sin^2\phi] + \pi^{1/2}\cos\theta\sin\phi[1 + \mathrm{erf}(S\cos\theta\sin\phi)]\} \tag{7b}$$

$$\tau_{iz} = (\tfrac{1}{2}\rho_\infty U^2/\pi^{1/2})\cos\theta\cos\phi\{(1/S)\exp[-S^2\cos^2\theta\sin^2\phi] + \pi^{1/2}\cos\theta\sin\phi[1 + \mathrm{erf}(S\cos\theta\sin\phi)]\} \tag{7c}$$

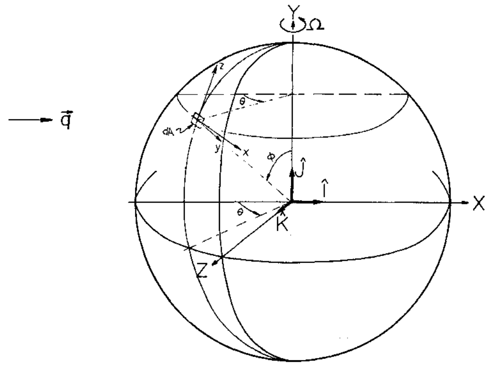

Fig. 1 Coordinate systems used for a free molecular flow over a rotating sphere.

Received December 7, 1971. The author wishes to acknowledge S. S. Shu for his guidance and help.

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where

$$F_u(\theta, \phi, s) = \sin\theta - s\sin\phi \tag{8}$$

$$S = U/(2RT_\infty)^{1/2} \tag{9a}$$

$$s = R_0\Omega/U \tag{9b}$$

$$\rho_\infty = mn_\infty \tag{10}$$

and erf(z) is the error function, defined as

$$\mathrm{erf}(z) = \frac{2}{\pi^{1/2}}\int_0^z e^{-t^2}\,dt \tag{11}$$

The momentum accommodation coefficients are defined as

$$\sigma \equiv (\tau_i - \tau_r)/(\tau_i - \tau_w) \tag{12}$$

$$\sigma' \equiv (p_i - p_r)/(p_i - p_w) \tag{13}$$

Here p_w and τ_w are the normal and tangential momentum components which would be carried away from a unit area per unit time by the diffusely reflected molecules as if they were at rest and in thermal equilibrium with the differential area; and p_r and τ_r are the normal and tangential components of reflected momentum flux. (τ_w can be proved to be zero.)

Based on the definitions given by Eqs. (12) and (13), and on Newton's second law, we can find the net pressure p, and shears τ_x, τ_y acting on the differential area

$$p = p_i + p_r = (2-\sigma')p_i + \sigma' p_w \tag{14a}$$

$$\tau_x = \tau_{ix} - \tau_{rx} = \sigma\tau_{ix} \tag{14b}$$

$$\tau_z = \tau_{iz} - \tau_{rz} = \sigma\tau_{iz} \tag{14c}$$

Using the foregoing definition for p_w and the steady-state condition that the number of incident molecules must be equal to the numbers of reflected molecules, p_w is obtained[6]

$$p_w = (\tfrac{1}{2}\rho_\infty U^2/2S^2)(T_w/T_\infty)^{1/2}\{\exp[-S^2\cos^2\theta\sin^2\phi] + \pi^{1/2}S\cos\theta\sin\phi[1+\mathrm{erf}(S\cos\theta\sin\phi)]\} \tag{15}$$

where T_w is the temperature of the differential area. We will assume T_w is constant throughout the sphere.

The forces acting on this differential area are

$$dF_x = \sigma\tau_{ix}\,dA \quad \text{in} \quad x \text{ direction} \tag{16a}$$

$$dF_y = [(2-\sigma')p_i + \sigma' p_w]\,dA \quad \text{in} \quad y \text{ direction} \tag{16b}$$

$$dF_z = \sigma\tau_{iz}\,dA \quad \text{in} \quad z \text{ direction} \tag{16c}$$

Drag and Lift Coefficients

The total drag acting on the sphere can be obtained by projecting dF_x, dF_y, and dF_z on the X direction, and integrating over the whole surface of the sphere. Using Eq. (2) and Eqs. (16), we may find the expression for the total drag

$$F_I = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\{[(2-\sigma')p_i + \sigma' p_w]\sin\phi\cos\theta + \sigma\tau_{ix}\sin\theta + \sigma\tau_{iz}\cos\phi\cos\theta\}R_0^2\sin\phi\,d\theta\,d\phi \tag{17}$$

Substituting Eqs. (7) and (15) into Eq. (17), we have, after performing the integration,

$$F_I = \tfrac{1}{2}\pi R_0^2\rho_\infty U^2\left\{\frac{(2-\sigma'+\sigma)}{2S^3}\left[\frac{(4S^4+4S^2-1)}{2S}\mathrm{erf}(S) + \frac{(2S^2+1)}{\pi^{1/2}}e^{-S^2}\right] + \frac{2}{3}\frac{\sigma'}{S}\pi^{1/2}\left(\frac{T_w}{T_\infty}\right)^{1/2}\right\} \tag{18}$$

$$C_D \equiv \frac{F_I}{(\tfrac{1}{2}\pi R_0^2\rho_\infty U^2)} = \frac{(2-\sigma'+\sigma)}{2S^3}\left[\frac{4S^4+4S^2-1}{2S}\mathrm{erf}(S) + \frac{2S^2+1}{\pi^{1/2}}e^{-S^2} + \frac{2}{3}\frac{\sigma'}{S}\pi^{1/2}\left(\frac{T_w}{T_\infty}\right)^{1/2}\right] \tag{19}$$

The rotation parameter s does not appear in Eqs. (18) and (19). The drag is not affected by the rotation of the sphere.

The integration over the whole sphere surface of the projections of dF_x, dF_y, and dF_z in the Y and Z directions will give the total lift components, F_J and F_K, in the Y and Z directions. Again, using Eq. (2) and Eqs. (16), we may obtain

$$F_J = 0 \tag{20}$$

$$F_K = -\tfrac{2}{3}\pi R_0^2\sigma s\rho_\infty U^2 \tag{21}$$

$$C_L \equiv F_K/(\tfrac{1}{2}\pi R_0^2\rho_\infty U^2) = -\tfrac{4}{3}\sigma s \tag{22}$$

The fact that F_J vanishes is physically obvious because of the symmetric properties of the geometry and the flowfield.

It should be noted that the magnitude of F_K is in direct proportion to $\rho_\infty\Gamma U$, where $\Gamma \equiv R_0\Omega$, and that F_K is acting in the negative Z direction. If we had an inviscid continuous fluid flowing over a rotating sphere as shown in Fig. (1), we would have a lift whose magnitude is also in direct proportion to $\rho_\infty\Gamma U$, but acting in the positive Z direction.

Limiting Case of Hypersonic Free Molecular Flow

In the extreme case where $S \to \infty$, (either $U \to \infty$, or $T_\infty \to 0$) we may consider the problem as that a stream of molecules with uniform velocity U strikes on the rotating sphere. If the temperature of the sphere is cold (i.e., $U/(RT_w)^{1/2} \to \infty$) and the reflection is purely diffusive with complete accommodation ($\sigma = \sigma' = 1$), the velocity of the reflected molecules is so small compared with U that it may be neglected. In this limiting case, the pressure and shears acting on the differential area are given as

$$p = \rho_\infty\bar{v}^2 \tag{23a}$$

$$\tau_x = \rho_\infty\bar{u}\bar{v} \tag{23b}$$

$$\tau_y = \rho_\infty\bar{w}\bar{v} \tag{23c}$$

where $\bar{u}$, $\bar{v}$, and $\bar{w}$ are given by Eqs. (5).

By the use of Eqs. (2), the X, Y, and Z components of aerodynamic force acting on the sphere can be expressed as

$$F_I = \int\int(\tau_x\sin\theta + p\cos\theta\sin\phi + \tau_z\cos\theta\cos\phi)R_0^2\sin\phi\,d\theta\,d\phi \tag{24a}$$

$$F_J = \int\int(-p\cos\phi + \tau_z\sin\phi)R_0^2\sin\phi\,d\theta\,d\phi \tag{24b}$$

$$F_K = \int\int(\tau_x\cos\theta - p_i\sin\theta\sin\phi - \tau_z\sin\theta\cos\phi)R_0^2\sin\phi\,d\theta\,d\phi \tag{24c}$$

where the domain of integration is $-\pi/2 \leqq \theta \leqq \pi/2$, $0 \leqq \phi \leqq \pi$. Substituting Eqs. (5) for $\bar{u}$, $\bar{v}$, and $\bar{w}$ in Eqs. (23), then substituting the results into Eqs. (24), we may obtain, after performing the integration,

$$F_I = \pi R_0^2\rho_\infty U^2 \tag{25a}$$

$$F_J = 0 \tag{25b}$$

$$F_K = -(2\pi/3)R_0^2 s\rho_\infty U^2 \tag{25c}$$

$$C_D \equiv F_I/(\tfrac{1}{2}\pi R_0^2\rho_\infty U^2) = 2 \tag{26a}$$

$$C_L \equiv F_K/(\tfrac{1}{2}\pi R_0^2\rho_\infty U^2) = -\tfrac{4}{3}s \tag{26b}$$

The drag F_I and the drag coefficient C_D can also be obtained from Eqs. (18) and (19) by taking the limit as $S \to \infty$. The same result can be found in the literature.[7] The lift F_K and lift coefficient C_L remain unchanged in this limiting case.

References

[1] Sentman, L. H., "Free Molecular Flow Theory and its Application to the Determination of Aerodynamic Forces," LMSC-448514, Oct. 1961, Lockheed Missiles & Space Co., Sunnyvale, Calif.

[2] Snow, R. M., "Aerodynamics in a Highly Attenuated Atmosphere," Rept. CM336, 1946, Johns Hopkins Univ., Baltimore, Md.

[3] Heineman, M., "Theory of Drag in Highly Rarefied Gases," *Communication on Pure and Applied Mathematics*, Vol. 1, 1948, pp. 259–273.

[4] Ashley, H. J., "Applications of the Theory of Free Molecule Flow to Aeronautics," *Journal of the Aeronautical Sciences*, Vol. 16, No. 2, 1949, pp. 95–104.

[5] Stalder, J. R. and Zurick, V. J., "Theoretical Aerodynamic Characteristics of Bodies in a Free-Molecular-Flow Field," TN 2423, 1951, NACA.

[6] Schaaf, S. A. and Chambre, P. L., *Flow of Rarefied Gases*, Princeton University Press, Princeton, N. J., 1961, pp. 11–18.

[7] Kogan, M. N., *Rarefied Gas Dynamics* (translated from Russian), Plenum Press, New York, 1968, pp. 414 and 415.